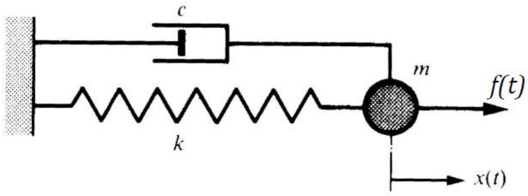
# Systèmes avec 1 DDL (degré de liberté)




| *Equation générale du mouvement (4 formes équivalents)* | | | |
|---|---|---|---|
| $m\ddot{x}+c\dot{x}+kx=f(t)$ | $\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=\frac{f(t)}{m}$ | $\ddot{x}+2\lambda\dot{x}+\omega_0^2x=\frac{f(t)}{m}$ | $\ddot{x}+2\eta\omega_0\dot{x}+\omega_0^2x=\frac{f(t)}{m}$ |
| $\omega_0=\sqrt{\frac{k}{m}}$ | $\eta=\frac{\lambda}{\omega_0}=\frac{c}{2m\omega_0}$ | $Libre \rightarrow f(t)=0$<br>$Conservatif \rightarrow c=0$ | $Forcé \rightarrow f(t)\neq 0$<br>$Dissipatif \rightarrow c\neq 0$ |

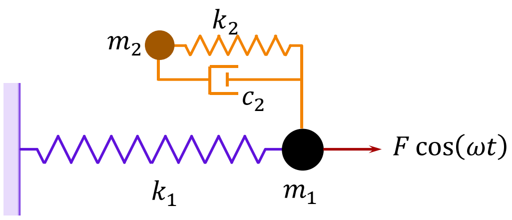
| *Système libre sur-amorti ($\eta>1$)* | | |
|---|---|---|
| $x(t)=Ae^{-r_1t}+Be^{-r_2t}$ | $r_1=\omega_0\left(\eta+\sqrt{\eta^2-1}\right)$ | $r_2=\omega_0\left(\eta-\sqrt{\eta^2-1}\right)$ |
| *Système libre sous-amorti ($\eta<1$)* | | |
| $x(t)=X_{libre}e^{-\lambda t}\cos(\omega_1t-\varphi_{libre})$ | $\omega_1=\omega_0\sqrt{1-\eta^2}$ | $x(t_{n+1})=x(t_n)e^{-\frac{2\pi\lambda}{\omega_1}}$ |
| *Puissance dissipée* $\overline{p(t)}=c\overline{\dot{x}^2}$ | Pour systèmes libres, $X_{libre}$ et $\varphi_{libre}$ dépendent sur les conditions initiales | |

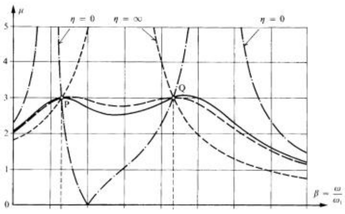
| *Système forcé – Régime permanente harmonique ($f(t)=F\cos(\omega t)$)* | | | | |
|---|---|---|---|---|
| $x(t)=X_{forcé}\cos\left(\omega t-\varphi_{forcé}\right)$ | $X_{forcé}=\frac{F}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\rightarrow X_{forcé}=\frac{\frac{F}{k}}{\sqrt{(1-\beta^2)^2+(2\eta\beta)^2}}=X_s\mu(\beta,\eta)$ | | | |
| | $\tan\left(\varphi_{forcé}\right)=\frac{c\omega}{k-m\omega^2}=\frac{2\eta\beta}{1-\beta^2}$ | | $\beta=\frac{\omega}{\omega_0}$ | |
| $x(t)=\mathrm{Re}\left(\underline{x}(t)\right)$ | $\underline{x}(t)=\frac{\underline{f}(t)}{k-m\omega^2+jc\omega}=\frac{\underline{f}(t)}{k}\underline{H}(\beta,\eta)=\frac{F}{k}\frac{e^{j(\omega t-\varphi)}}{\sqrt{(1-\beta^2)^2+(2\eta\beta)^2}}$ | | $\underline{H}(\beta,\eta)=\frac{1}{1-\beta^2+j2\eta\beta}$ | |
| Résonance en amplitude | Résonance en vitesse et phase | | Résonance en accélération | |
| $\omega_2=\omega_0\sqrt{1-2\eta^2}$ | $\omega_0$ | | $\omega_3=\omega_0/\sqrt{1-2\eta^2}$ | |
| $\mu(\beta_2,\eta)=(2\eta)^{-1}(1-\eta^2)^{-1/2}$ | $\mu(\beta_0,\eta)=(2\eta)^{-1}$ | | $\mu(\beta_3,\eta)=(1-2\eta^2)(2\eta)^{-1}(1-\eta^2)^{-1/2}$ | |
| *Puissance dissipée :* $\overline{p(t)}=c\overline{\dot{x}^2}=\overline{f(t)\dot{x}(t)}=\frac{F^2}{m\omega_0}\frac{\eta\beta^2}{(1-\beta^2)^2+(2\eta\beta)^2}$ | | | | |

| *Système forcé – Régime permanente périodique ($f(t)=f(t+T)=f(t+2\pi/\omega)$)* | | | |
|---|---|---|---|
| $f(t)=\frac{F_0}{2}+\sum_{n=1}^{\infty}(A_n\cos(n\omega t)+B_n\sin(n\omega t))$ | $A_n=\frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt$ | $B_n=\frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$ | |
| $f(t)=\frac{F_0}{2}+\sum_{n=1}^{\infty}F_n\cos(n\omega t-\psi_n)$ | $F_n=\sqrt{A_n^2+B_n^2}$ | $\tan(\psi_n)=B_n/A_n$ | |
| $x(t)=\frac{F_0}{2k}+\sum_{n=1}^{\infty}X_n\cos(n\omega t-\psi_n-\varphi_n)$ | $\mu_n=\frac{1}{\sqrt{(1-n^2\beta^2)^2+(2\eta n\beta)^2}}$ | $\tan(\varphi_n)=\frac{2\eta n\beta}{1-n^2\beta^2}$ | |
| $\underline{f}(t)=D_0+\sum_{n=1}^{\infty}\underline{D}_ne^{jn\omega t}$ | $\underline{x}(t)=\frac{D_0}{k}+\sum_{n=1}^{\infty}\frac{\underline{H}_n}{k}\underline{D}_ne^{jn\omega t}$ | $\underline{D}_n=\frac{2}{T}\int_0^T f(t)e^{-jn\omega t}\,dt$ | $\underline{H}_n=\frac{1}{1-n^2\beta^2+j2\eta n\beta}$ |

## Amortisseur de Frahm



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|---|---|---|---|---|
| $x_1(t) = Re(\underline{x_1}) \rightarrow \underline{x_1} = \underline{A_1}e^{j\omega t} = X_1 e^{-\varphi_1}e^{j\omega t}$ | $\underline{A_1} = F\dfrac{(k_2-\omega^2 m_2)+j\omega c_2}{(k_1-\omega^2 m_1)(k_2-\omega^2 m_2)-\omega^2 m_2 k_2 + j\omega c_2(k_1-\omega^2(m_1+m_2))}$ | | | |
| $\mu_1^2 = \dfrac{X_1^2}{X_{1s}^2} = \dfrac{4\eta^2\beta^2+(\beta^2-\alpha^2)^2}{4\eta^2\beta^2(\beta^2(1+\varepsilon)-1)^2+(\varepsilon\alpha^2\beta^2-(\beta^2-1)(\beta^2-\alpha^2))^2}$ | $\varepsilon = \dfrac{m_2}{m_1}$ | $\alpha = \dfrac{\omega_2}{\omega_1}$ | $\omega_1 = \sqrt{\dfrac{k_1}{m_1}}$ | $\beta = \dfrac{\omega}{\omega_1}$ |
| $\mu_2^2 = \dfrac{X_2^2}{X_{1s}^2} = \dfrac{4\eta^2\beta^2+\alpha^4}{4\eta^2\beta^2(\beta^2(1+\varepsilon)-1)^2+(\varepsilon\alpha^2\beta^2-(\beta^2-1)(\beta^2-\alpha^2))^2}$ | $\eta = \dfrac{c_2}{2m_2\omega_1}$ | | $\omega_2 = \sqrt{\dfrac{k_2}{m_2}}$ | $X_{1s} = \dfrac{F}{k_1}$ |
| *Optimisé pour limiter mouvement de $m_1$ sur une large plage fréquentielle :* | | | | |
| $\alpha = \dfrac{1}{1+\varepsilon} = \dfrac{m_1}{m_1+m_2}$ | $\beta_P^2 = \dfrac{1}{1+\varepsilon}\left(1-\sqrt{\dfrac{\varepsilon}{2+\varepsilon}}\right);\ \beta_Q^2 = \dfrac{1}{1+\varepsilon}\left(1+\sqrt{\dfrac{\varepsilon}{2+\varepsilon}}\right)$ | $\mu_{1,P} = \mu_{1,Q} = \sqrt{\dfrac{2+\varepsilon}{\varepsilon}}$ | *$\eta$ pour $\mu_1$ maximum à P ou Q* | |

## Systèmes avec n DDL

$$\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn}\end{pmatrix}\begin{pmatrix}\ddot{x}_1\\ \vdots \\ \ddot{x}_n\end{pmatrix} + \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn}\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \vdots \\ \dot{x}_n\end{pmatrix} + \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix}f_1\\ \vdots \\ f_n\end{pmatrix} \qquad \boldsymbol{M}\ddot{\vec{x}} + \boldsymbol{C}\dot{\vec{x}} + \boldsymbol{K}\vec{x} = \vec{f}$$

| | | |
|---|---|---|
| *Systèmes satisfaisant Caughey :* | $\boldsymbol{C}\cdot\boldsymbol{M}^{-1}\cdot\boldsymbol{K} = \boldsymbol{K}\cdot\boldsymbol{M}^{-1}\cdot\boldsymbol{C}$ | |
| *Matrice noyau :* $\boldsymbol{A} = \boldsymbol{M}^{-1}\boldsymbol{K}$ | $\det(\boldsymbol{A}-\delta\mathbb{I}) = 0 \rightarrow \{\delta_1,\cdots,\delta_n\}$ *valeurs propres* $\rightarrow \omega_{0_i} = \sqrt{\delta_i}$ *freqs propres* | |
| | $\boldsymbol{A}\cdot\vec{v}_i = \omega_{0_i}^2\cdot\vec{v}_i \rightarrow \vec{v}_i$ *vecteurs propres* $\rightarrow \boldsymbol{B} = (\vec{v}_1 \ \ \ldots \ \ \vec{v}_n)$ | |
| *Quotient de Rayleigh :* $R(\vec{v}) = \dfrac{\vec{v}^T\boldsymbol{K}\vec{v}}{\vec{v}^T\boldsymbol{M}\vec{v}}$ | $R(\vec{v}_i) = \dfrac{\vec{v}_i^T\boldsymbol{K}\vec{v}_i}{\vec{v}_i^T\boldsymbol{M}\vec{v}_i} = \dfrac{k_i}{m_i} = \omega_{0_i}^2$ | $R(\text{proche de } \vec{v}_i) \approx \omega_{0_i}^2$ |
| *Relation entre coordonnées modales et réelles :* $\vec{x} = \boldsymbol{B}\cdot\vec{q}$ | $\boldsymbol{M^0} = \boldsymbol{B^T M B};\ \boldsymbol{C^0} = \boldsymbol{B^T C B};\ \boldsymbol{K^0} = \boldsymbol{B^T K B}$ | |
| $\boldsymbol{\Omega_0^2} = \boldsymbol{M^0}^{-1}\boldsymbol{K^0} = \boldsymbol{B}^{-1}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{B}$ | $\boldsymbol{2\Lambda} = \boldsymbol{M^0}^{-1}\boldsymbol{C^0} = \boldsymbol{B}^{-1}\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{B}$ | $\boldsymbol{\Omega_0^2}$ *et* $\boldsymbol{2\Lambda}$ *diagonales avec* $\omega_{0_i}^2$ *et* $2\lambda_i$ |
| *Système découplé, résoudre n fois 1 DDL :* | $\boldsymbol{M^0}\ddot{\vec{q}} + \boldsymbol{C^0}\dot{\vec{q}} + \boldsymbol{K^0}\vec{q} = \vec{f}^0 = \boldsymbol{B^T}\vec{f}$ | $\ddot{\vec{q}} + \boldsymbol{2\Lambda}\dot{\vec{q}} + \boldsymbol{\Omega_0^2}\vec{q} = \boldsymbol{M^0}^{-1}\vec{f}^0$ |
| *Conservatif :* | $\vec{x} = \displaystyle\sum_{i=1}^{n}\frac{\vec{v}_i}{m_i^0\omega_{0_i}}\left(\int_0^t f_i^0(t-\tau)\sin(\omega_{0_i}\tau)\,d\tau + \vec{v}_i^T\boldsymbol{M}\,\dot{\vec{x}}_{ini}\sin(\omega_{0_i}t) + \omega_{0_i}\vec{v}_i^T\boldsymbol{M}\,\vec{x}_{ini}\cos(\omega_{0_i}t)\right)$ | |
| *Forcé harmonique :* | $\underline{\vec{X}}(j\omega) = \displaystyle\sum_{i=1}^{n}\vec{v}_i\underline{Q_i}(j\omega) = \left[\sum_{i=1}^{n}\frac{\vec{v}_i\otimes\vec{v}_i}{m_{0i}(-\omega^2+2j\eta_i\omega_{0i}\omega+\omega_{0i}^2)}\right]\underline{\vec{F}}(j\omega) = \underline{H}(j\omega)\underline{\vec{F}}(j\omega)$ | |

## Systèmes Continus

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|---|---|---|---|
| *Vibration (y) de Cordes :* <br> *(σ contrainte, ρ densité)* | $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\sigma}{\rho}\dfrac{\partial^2 y}{\partial x^2}$ | *Vibration (u) de Barres :* <br> *(E module de Young, ρ densité)* | $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{E}{\rho}\dfrac{\partial^2 u}{\partial x^2}$ |
| *Vibration (φ) en Torsion :* <br> *(G module de glissement, ρ densité)* | $\dfrac{\partial^2 \varphi}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{G}{\rho}\dfrac{\partial^2 \varphi}{\partial x^2}$ | *Solution :* | $\left(A_n\cos\left(\dfrac{\omega_n}{a}x\right) + B_n\sin\left(\dfrac{\omega_n}{a}x\right)\right)\cos(\omega_n t)$ |
| *Vibration (y) de poutres droites :* <br> *(I moment d'inertie, A aire de la section )* | $\dfrac{\partial^2 y}{\partial t^2} = -\dfrac{EI}{\rho A}\dfrac{\partial^4 y}{\partial x^4}$ | | |
| *Solution :* | $y_n(x,t) = \left(A_n\cos\left(\dfrac{\alpha_n}{L}x\right) + B_n\sin\left(\dfrac{\alpha_n}{L}x\right) + C_n\cosh\left(\dfrac{\alpha_n}{L}x\right) + D_n\sinh\left(\dfrac{\alpha_n}{L}x\right)\right)\cos(\omega_n t)$ | $\left(\dfrac{\alpha_n}{L}\right)^4\dfrac{EI}{\rho A} = \omega_n^2$ | |